CERN–TH/95–14
hep-th/9502007

Anisotropic space-times in homogeneous string cosmology

Nikolaos A. Batakis * †

Theory Division, CERN
CH–1211 Geneva 23, Switzerland

and

Alexandros A. Kehagias ‡

Phys. Dept., National Technical Univ.
GR–15773 Zografou Athens, Greece

ABSTRACT

The dynamics of the early universe may have been profoundly influenced by spatial anisotropies. A search for such backgrounds in the context of string cosmology has uncovered the existence of an entire class of (spatially) homogeneous but not necessarily isotropic space-times, analogous to the class of Bianchi-types in general relativity. Configurations with vanishing cosmological constant but with non-vanishing dilaton and antisymmetric field are explicitly found for all types. This is a new class of solutions, whose isotropy limits reproduce all known and, further, all possible FRW-type of models in the string-cosmology context considered. There is always an initial singularity and no inflation. Other features of the general solutions, including their behaviour

CERN–TH/95–14
January 1995

*Permanent address: Department of Physics, University of Ioannina, GR–45100 Ioannina, Greece
†e–mail address: batakis@surya20.cern.ch
‡e-mail address: kehagias@si.kun.nl

1 Introduction

There are several compelling reasons (to be reviewed shortly) which indicate that the dynamics of the early universe may have been profoundly influenced by the presence of spatial anisotropies just below the Planck or string scale [1]. It follows that the assumption of a Friedmann-Robertson-Walker (FRW) behaviour may be hindering important aspects of the early dynamics and other cosmological issues traceable to that era. Thus motivated we have relaxed the requirment of spatial isotropy in the FRW-type of symmetry in order to study (spatially) homogeneous models as background space-times for string theory [2],[3]. We have found that the one-loop beta-function equations in four dimensions admit an entire class of solutions representing homogeneous but not necessarily isotropic space-times. This class, which will be explicitly presented in this paper, may be considered as the string-cosmology counterpart of what is known in general relativity as the class of vacuum Bianchi-type cosmologies [4],[5]. Without being excessively symmetric (they possess three Killing vectors compared to the six of the FRW and the ten of the Minkowski space-time), the Bianchi-type space-times allow for their full classification and a rather elegant mathematical treatment. Even more important seems the fact that any given FRW behaviour may be viewed as the isotropic limit of some homogeneous spacetime. It follows that the study of such spacetimes (in the context of what one might appropriately call 'homogeneous string cosmology') may offer us tractable and promising models for understanding the dynamics of anisotropy and its potentially crucial rôle in the early universe, until the attainment of the observed state of isotropy.

To establish notation we recall that the one-loop beta-function equations for conformal invariance are [6]

$$R_{\mu\nu} - \frac{1}{4}H^2_{\mu\nu} - \nabla_\mu\nabla_\nu\phi = 0, \tag{1}$$

$$\nabla^2(e^\phi H_{\mu\nu\lambda}) = 0, \tag{2}$$

$$-R + \frac{1}{12}H^2 + 2\nabla^2\phi + (\partial_\mu\phi)^2 + \Lambda = 0, \tag{3}$$

where the contractions $H^2_{\mu\nu} = H_{\mu\kappa\lambda}H_\nu{}^{\kappa\lambda}$, $H^2 = H_{\mu\nu\lambda}H^{\mu\nu\lambda}$ involve the totally antisymmetric field strenght $H_{\mu\nu\lambda}$, defined in terms of the potential $B_{\mu\nu}$ as

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}. \tag{4}$$

In a 4-dimensional context, the set (1–3) can be derived from an effective action of the form

$$S_{eff} = \int d^4x\sqrt{-g}e^\phi(R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \partial_\mu\phi\partial^\mu\phi - \Lambda). \tag{5}$$

The constant Λ, directly related to the charge deficit δc in the original theory, is proportional to $D-26$ or $D-10$ in the bosonic or heterotic string respectively. It therefore

vanishes at critical dimensions or otherwise may be neglected at sufficiently large curvatures R or kinetic energies $(\nabla\phi)^2$, H^2 for the dilaton and B field. The action (5), already in the so-called sigma frame, may be recast as

$$S_{eff} = \int d^4x\sqrt{-\tilde{g}}(\tilde{R} - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}e^{2\phi} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \Lambda e^{-\phi}). \tag{6}$$

wherein the metric is conformally related to the previous one by

$$\tilde{g}_{\mu\nu} = e^\phi g_{\mu\nu} \tag{7}$$

and it is being utilized to raise indices etc in this more conventional 'Einstein frame'. Eq. (1) resumes then a typical general relativistic form as

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = \kappa^2(T^{(\phi)}_{\mu\nu} + T^{(H)}_{\mu\nu}) \tag{8}$$

with

$$\begin{aligned} \kappa^2 T^{(\phi)}_{\mu\nu} &= \frac{1}{2}(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2\tilde{g}_{\mu\nu} - \Lambda e^{-\phi}\tilde{g}_{\mu\nu})\,, \\ \kappa^2 T^{(H)}_{\mu\nu} &= \frac{1}{4}e^{2\phi}(H_{\mu\kappa\lambda}H_\nu{}^{\kappa\lambda} - \frac{1}{6}H^2\tilde{g}_{\mu\nu})\,, \end{aligned} \tag{9}$$

the energy-momentum tensor for ϕ and H, while the remaining Eqs. (2, 3) become

$$\begin{aligned} \tilde{\nabla}^2(e^\phi H_{\mu\nu\lambda}) &= 0, \\ -\frac{1}{6}e^{2\phi}H^2 + \tilde{\nabla}^2\phi + \Lambda e^{-\phi} &= 0. \end{aligned} \tag{10}$$

From experience in general relativity we know that searching for the general solution to the system of Eqs. (1–3) or equivalently (8–10) is a hopeless and perhaps meaningless enterprise. Without any underlying symmetry [7],[8] or other physical input, such a coupled system of non-linear differential equations can supply an immence plethora of configurations representing chaotic or other intractable excitations, which would have little or nothing to do with the real world. On the other hand, several interesting cosmological solutions have been constructed explicitly either by directly solving the background field equations or by exploiting conformal theory techniques [2],[3], [9]–[14]. As far as we are aware, a classification of solutions has yet to appear in the literature.

The structure of the rest of this paper is as follows. In the next section we proceed with an introduction to homogeneous string cosmology and the reasons which motivate its study. In section 3 we write down the background field equations in the sigma frame, to be followed by explicit solutions for each Bianchi type, presented in section 4. These reults are briefly discussed in the last section and summarized in a Table. The latter is preceeded by an Appendix with definitions and proceedures necessary for the reproduction of our results.

2 Homogeneous string cosmology

By relaxing the requirement of isotropy made in the FRW-type of models, one is led to the study of (spatially) homogeneous but not necessarily isotropic space-times M^4 [1],[15],[16]. The arguments which motivate this study may be summarized as follows.

1. The generally claimed isotropy of the universe is deduced from fairly reliable observations and measurments, including those on the cosmic microwave background, which are by far the most accurate ones in cosmology. However, all these results have been established for times subsequent to the era at which the universe became transparent to the electromagnetic radiation. Their extrapolation to earlier times and, in particular, near the Planck or string scale is totally unfounded.

2. Statistical flactuation in the FRW models or otherwise small perturbations of the initial conditions either grow or otherwise do not smooth-out sufficiently fast to conform with the observed state of the universe, thus giving rise to a host of paradoxes such as the flatness and the horizon problems. It follows that in any scenario (such as inflation) to resolve these problems, the observed FRW behaviour should emerge as a consequence rather than taken as starting assumption.

3. The kinetic and potential energy of the gravitational field which may be attributed to the presence of anisotropy could reach the same order of magnitute or even exeed the energy attributed to any other field present in the effective action (6) at some time in the early evolution. Hence, the dynamics of the universe at that time could be profoundly distorted if one neglected that contribution.

4. The presence and significance of an initial singularity is hardly questioned as based on both observational as well as theoretical grounds. However, the dynamics and structure of such an initial state as predicted by the FRW behaviour may change profoundly in the presence of anisotropy.

As in conventional general relativity, homogeneous backgrounds in string cosmology may be defined as those 4-dimensional space-time manifolds which admit an r-parameter group of isometries G_r whose orbits in M^4 are 3-dimensional space-like hypersurfaces Σ^3. The latter are precisely the hypersurfaces of homogeneity on which a G_3 subgroup of G_r acts transitively. Clearly, $3 \leq r \leq 6$, so that any remaining symmetry (and the corresponding independent Killing vectors) must generate the (r-3)-dimensional isotropy subgroup of G_r. Since there is no two-dimensional rotation group, (r-3) can only have the values $0, 1, 3$, with the last one associated with the maximal FRW-type of symmetry. The action of G_3 is almost always simply transitive on its orbits. The exception is been realized in the Kantowski-Sachs type of metric, in which G_3 is actually acting on

2-dimensional spacelike surfaces of maximal symmetry. In the typical case which, as mentioned, involves a simply-transitive G_3, the most general metric may be written as

$$ds^2 = -dt^2 + g_{0i}dt\sigma^i + g_{ij}\sigma^i\sigma^j, \tag{11}$$

where the metric coefficients may be functions of the time t only and $\{\sigma^i,\, i = 1, 2, 3\}$ is a basis of 1-forms, invariant under the left action of G_3.

Our study of Bianchi-type models in string cosmology will be made under the assumption of a vanishing cosmological constant. We will also restrict ourselves to metrics which are diagonal in the non-holonomic frame employed in Eq. (11) namely are expressible as

$$ds^2 = -dt^2 + a_1(t)^2(\sigma^1)^2 + a_2(t)^2(\sigma^2)^2 + a_3(t)^2(\sigma^3)^2 \tag{12}$$

All possible isometry groups G_3 of the metric (12) are known and have been fully classified. Each one of the existing nine Bianchi types is identified by the corresponding set of group-structure constants C^i_{jk} which define the relation

$$d\sigma^i = -\frac{1}{2}C^i_{jk}\sigma^j \wedge \sigma^k \tag{13}$$

or, equivalently,

$$[\xi_i, \xi_j] = -C^k_{ij}\xi_k, \tag{14}$$

where the set of the three independent Killing vectors ξ_i forms a basis dual to $\{\sigma^i\}$.

The simplest case is that of a Bianchi-type I model in which G_3 is just the abelian tranlation group T_3 and the hypersurfaces of homogeneity Σ^3 are Euclidean 3-dimensional sheets. At the other end of the Bianchi classification we find the type-IX models for $G_3 = SO(3)$, and Σ^3 sections with S^3 topology. We also note that Bianchi-type VI_h space-times involve an entire one-parameter family of G_3 groups, parametrized by h which takes all real values exept 0,1. Likewise, a one-parameter family of G_3 groups is involved in the Bianchi-type VII_h case, here with $-2 < h < 2$. The situation is much simpler with the FRW models, which of course are fully classified by $k = 0, -1, +1$ and also have an $SO(3)$-isotropy group in addition to their G_3. All these G_6 space-times are typically realized by Bianchi-types I (and VII_0), V,IX respectively. All Bianchi-types (as summarized in the Table) have been further classified as being of class A if their adjoint representation is traceless, otherwise they are of class B.

3 The background field equations

In the next section we will examine each Bianchi type separately. However, before doing that it will be usefull to present in this section features of the background field equations which are common to all types.

3.1 The dilaton field and the totally antisymetric field strength

To respect homogeneity, any scalar such as the dilaton ϕ must be a constant on each hypersurface Σ_3 so that, in M^4, ϕ can at most be a function of the time t. On similar grounds we may express the 3-form H as

$$H = A\sigma^1 \wedge \sigma^2 \wedge \sigma^3, \tag{15}$$

which, with A considered as at most a function of t only, is equivalent to the usual trading of H for a time dependent axion field in 4D. Here however, in view of the $dH = 0$ requirment, A must be a constant not only on Σ^3 but everywhere in M^4. It is reminded however that in conventional (namely holonomic) coordinates, the components of the same H will certainly depend on t and, in general, on the spatial coordinates as well. It follows that Eq. (2) is then automatically satisfied for any time-dependent configuration of the dilaton field. With a dot standing for d/dt, Eq. (3) may now be expressed as

$$\ddot{\phi} + (\dot{\phi})^2 + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{1}{6}(\frac{A}{a^3})^2 = 0. \tag{16}$$

where we have introduced the radius a defined by

$$a^3 = a_1 a_2 a_3, \tag{17}$$

so that $\dot{a}/a$ is the mean Hubble constant of any comoving volume element (the latter is always proportional to a^3). In terms of the new time coordinate τ defined by

$$d\tau = a^{-3}e^{-\phi}dt, \tag{18}$$

and with a prime standing for $d/d\tau$ we may re-express Eq. (16) as

$$\phi'' + A^2 e^{2\phi} = 0. \tag{19}$$

The general solution of the above equation may be expressed as

$$e^{-\phi} = \cosh(N\tau) + \sqrt{1 - \frac{A^2}{N^2}}\sinh(N\tau), \tag{20}$$

where N is a constant, while a second constant of integration has been absorbed to fix the origin of ϕ. In an orthonormal frame $\{\omega^\mu\}$ (cf. Appendix), the energy-momentum tensors (9) for the ϕ and H fields have the form of

$$\begin{aligned}
\kappa^2 T^{(\phi)}_{\mu\nu} &= \frac{1}{4}a^{-6}e^{-3\phi}(\frac{d\phi}{d\tau})^2 diag(1,1,1,1), \\
\kappa^2 T^{(H)}_{\mu\nu} &= \frac{1}{4}A^2 a^{-6} diag(1,1,1,1).
\end{aligned} \tag{21}$$

These expressions will be briefly recalled in the last section.

3.2 The gravitational field equations

As outlined in the Appendix, the set of Eqs. (1) involves three second order differential equations for the radii $a_i(t)$ which are of the form

$$(\ln a_i^2 e^{\phi})'' + 2V_i = 0,\ \ (i = 1, 2, 3) \tag{22}$$

Although in the sigma-model frame, these equations are expressed more concisely in terms of the variables $a_i^2 e^{\phi}$, which, according to (7), are just the metric coefficients $\tilde{a}_i^2$ in the Einstein frame. Each 'potential' V_i is generally dependent on all three radii $\tilde{a}_j$. Eqs. (22) are in fact the (ii) components of the set (1). Their solutions $a_i(t)$ are subject to

i) The initial value equation

$$\sum_{i<j} (\ln a_i^2 e^{\phi})'(\ln a_j^2 e^{\phi})' + 2\sum V_i = A^2, \tag{23}$$

which is essentialy the (00) equation in the set (1).

ii) A set of constraint equations which are of the form

$$R_{\mu\nu} = 0,\ \ (\mu \neq \nu). \tag{24}$$

and are due to the emergence of non-vanishing off-diagonal components of the Ricci tensor. For the type of metrics considered here, these arise only in the case of class-B types.

4 Explicit solutions

In this section we examine the background field equations and solve them for each Bianchi-type separately. We recall that the expressions (20,15) which specify the dilaton and the H-field hold invariably for all Bianchi types. The results presented here will be further discussed in the last section.

4.1 Type I

As mentioned already, in Type-I models the Σ^3 hypersurfaces are 3-dimensional flat Euclidean sheets generated by the group of abelian translations T_3. The set (22) is simply

$$(\ln a_i^2 e^{\phi})'' = 0, \tag{25}$$

whose general solution is

$$a_i^2 e^{\phi} = L_i e^{p_i \tau}. \tag{26}$$

There are no constraint equations but the initial value equation (23) imposes on the constants p_i the Kasner-like restriction

$$\sum_{i<j} p_i p_j = A^2. \tag{27}$$

When $A = 0$ the above solution reduces to the one found in [2]. When the isometry group is enlarged to $T_3 \times SO(3)$, namely at the isotropic limit attained at $a_1 = a_2 = a_3$, the above solution reduces to the one given in [11].

4.2 Type II

Here Eqs. (22) are expressed as

$$\begin{aligned}
(\ln a_1^2 e^{\phi})'' + a_1^4 e^{2\phi} &= 0, \\
(\ln a_2^2 e^{\phi})'' - a_1^4 e^{2\phi} &= 0, \\
(\ln a_3^2 e^{\phi})'' - a_1^4 e^{2\phi} &= 0,
\end{aligned} \tag{28}$$

which admit the general solution

$$\begin{aligned}
a_1^2 e^{\phi} &= \frac{p_1}{\cosh(p_1 \tau)}, \\
a_2^2 e^{\phi} &= L_2^2 \cosh(p_1 \tau) e^{p_2 \tau}, \\
a_3^2 e^{\phi} &= L_3^2 \cosh(p_1 \tau) e^{p_3 \tau}.
\end{aligned} \tag{29}$$

There are no constraint equations but Eq. (23) subjects the constants p_i to the restriction

$$p_2 p_3 - p_1^2 = A^2. \tag{30}$$

4.3 Type III

This is a class B space-time subject to the constraint equation

$$(\ln \frac{a_3}{a_1})' = 0 \tag{31}$$

The set of Eqs. (22) can be written as

$$\begin{aligned}
(\ln a_1^2 e^{\phi})'' - 2(a_1 a_2 e^{\phi})^2 &= 0, \\
(\ln a_2^2 e^{\phi})'' &= 0,
\end{aligned} \tag{32}$$

where without loss of generallity we have assumed that $a_1 = a_3$. The general solution to the above system is

$$\begin{aligned}
a_1^2 e^{\phi} &= \frac{p_1}{\sinh^2(p_1 \tau)} e^{p_2 \tau}, \\
a_2^2 e^{\phi} &= p_1 e^{-p_2 \tau},
\end{aligned} \tag{33}$$

with the integration constants subject to the restriction

$$4p_1^2 - p_2^2 = A^2 \tag{34}$$

imposed by Eq. (23).

4.4 Type IV

In this (class B) case the constraint equations (24) give

$$\begin{aligned} \frac{a_1}{a_2 a_3^2} &= 0, \\ \frac{1}{a_3}(\ln \frac{a_1 a_2}{a_3^2})' &= 0 \end{aligned} \tag{35}$$

so that, within the present context, all solutions are singular everywhere in M^4.

4.5 Type V

The constraint equations (24) impose in this case the restriction (q_1 is a constant)

$$a_1^2 = q_1 a_2 a_3. \tag{36}$$

The field equations (22) are

$$\begin{aligned} (\ln a_1^2 e^{\phi})'' - 4(a_2 a_3 e^{\phi})^2 &= 0, \\ (\ln a_2^2 e^{\phi})'' - 4(a_2 a_3 e^{\phi})^2 &= 0, \\ (\ln a_3^2 e^{\phi})'' - 4(a_2 a_3 e^{\phi})^2 &= 0. \end{aligned} \tag{37}$$

It follows that the most general solution which satisfies the above constraint is

$$\begin{aligned} a_1^2 e^{\phi} &= \frac{q_1 p_1}{2\sinh(p_1 \tau)}, \\ a_2^2 e^{\phi} &= \frac{q_2 p_1}{2\sinh(p_1 \tau)} e^{p_2 \tau}, \\ a_3^2 e^{\phi} &= \frac{p_1}{2 q_2 \sinh(p_1 \tau)} e^{-p_2 \tau}. \end{aligned} \tag{38}$$

The constants p_1, p_2 are subject to the restriction

$$3p_1^2 - p_2^2 = A^2 \tag{39}$$

imposed by Eq. (23). When the isometry group is enlarged to a G_6 the above solution reduces to its isotropic limit attained at $p_1 = A/\sqrt{3}$ and $q_1 = q_2 = 1$. This represents an open (k=-1) FRW model with radius

$$a^2 e^{\phi} = \frac{A}{2\sqrt{3}\sinh(\frac{A}{\sqrt{3}}\tau)}, \tag{40}$$

which has been already discussed in the quoted literature.

4.6 Type VI_h

The G_3 involved in this type of models is actually a continuous 1-parameter family of groups parametrized by h, with the values $h \neq 0, 1$ typically excluded as giving rise to Bianchi types III and V respectively. To facilitate calculations we will assume for the moment that we also have $h \neq -1$ and present the Bianchi-type VI_{-1} space-time separately in the next subsection. These models are subject by Eqs. (24) to the constraint

$$\frac{a_2^h a_3}{a_1^{h+1}} = \frac{q_2^{h-1}}{q_1^{h+1}}, \tag{41}$$

where the rhs has been expressed in terms of the numerical constants q_1, q_2, introduced for later convenience. The field equations (22) are

$$\begin{aligned} (\ln a_1^2 e^{\phi})'' - 2(h^2+1)(a_2 a_3 e^{\phi})^2 &= 0, \\ (\ln a_2^2 e^{\phi})'' - 2h(h+1)(a_2 a_3 e^{\phi})^2 &= 0, \\ (\ln a_3^2 e^{\phi})'' - 2(h+1)(a_2 a_3 e^{\phi})^2 &= 0. \end{aligned} \tag{42}$$

This system can be fully integrated and the most general solution which satisfies the above constraint is given by

$$\begin{aligned} a_1^2 e^{\phi} &= q_1^2 (\frac{p_1}{h+1})^{\frac{2(h^2+1)}{(h+1)^2}} \sinh(p_1 \tau)^{-\frac{2(h^2+1)}{(h+1)^2}} e^{\frac{(h-1)}{h+1} p_2 \tau}, \\ a_2^2 e^{\phi} &= q_2^2 (\frac{p_1}{h+1})^{\frac{2h}{h+1}} \sinh(p_1 \tau)^{-\frac{2h}{h+1}} e^{p_2 \tau}, \\ a_3^2 e^{\phi} &= q_2^{-2} (\frac{p_1}{h+1})^{\frac{2}{h+1}} \sinh(p_1 \tau)^{-\frac{2}{h+1}} e^{-p_2 \tau}. \end{aligned} \tag{43}$$

The constants p_1, p_2 are subject to the restriction

$$\frac{4(h^2+h+1)}{(h+1)^2} p_1^2 - p_2^2 = A^2, \tag{44}$$

as required by Eq. (23). We observe however that, in this particular type, the restriction imposed by the initial value equation can be evaded. In other words, the constants p_1, p_2 may be chosen arbitrarily, but at the expense of fine-tunning for specifically chosen group parameters h, so that Eq.(44) is satisfied.

4.7 Type VI_{-1}

For $h = -1$ the constraint equation (41) may without loss of generality be written as $a_2 = a_3$ so that Eqs.(42) reduce to

$$\begin{aligned} (\ln a_1^2 e^{\phi})'' - 4 a_2^4 e^{2\phi} &= 0, \\ (\ln a_2^2 e^{\phi})'' &= 0 \end{aligned} \tag{45}$$

The general solution to this system is

$$\begin{aligned} a_1^2 e^{\phi} &= q_1^2 p_1 exp(q_2^4 e^{2p_2\tau}) e^{p_1\tau}, \\ a_2^2 e^{\phi} &= q_2^2 p_2 e^{p_2\tau}, \end{aligned} \tag{46}$$

with the constants p_1, p_2 subject to the restriction

$$p_2^2 + 2p_1 p_2 = A^2,$$

coming from Eq. (23).

4.8 Type VII_h

In this case the constraints imposed by (24) are

$$\begin{aligned} \frac{h}{a_3}(\ln \frac{a_2}{a_3})' &= 0, \\ \frac{h a_2}{a_1 a_3} &= 0 \end{aligned} \tag{47}$$

We observe that all solutions are singular everywhere unless $h = 0$. In the latter case (which involves space-times of class A) the field equations (22) may be expressed as

$$\begin{aligned} (\ln a_1^2 e^{\phi})'' + (a_1^4 - a_2^4) e^{2\phi} &= 0, \\ (\ln a_2^2 e^{\phi})'' + (a_2^4 - a_1^4) e^{2\phi} &= 0, \\ (\ln a_3^2 e^{\phi})'' - (a_1^2 - a_2^2)^2 e^{2\phi} &= 0. \end{aligned} \tag{48}$$

We have not being able to write down the general solution to the above equations in closed form. We observe, however, that by enlarging the isotropy group to a G_4 (namely with $a_1 = a_2$) the above equations reduce to the Bianchi-type I set so that the corresponding solutions may be recovered from there.

4.9 Type VIII

For space-times in this type there are no constraint equations and Eqs. (22) are written as

$$\begin{aligned} (\ln a_1^2 e^{\phi})'' + (a_1^4 - (a_2^2 + a_3^2)^2) e^{2\phi} &= 0, \\ (\ln a_2^2 e^{\phi})'' + (a_2^4 - (a_3^2 - a_1^2)^2) e^{2\phi} &= 0, \\ (\ln a_3^2 e^{\phi})'' + (a_3^4 - (a_1^2 + a_2^2)^2) e^{2\phi} &= 0, \end{aligned} \tag{49}$$

A general analytic solution to the above system seems attainable only after enlargment of the isotropy group to a G_4 (e.g., with $a_1 = a_3$). We then have

$$\begin{aligned} a_1^2 e^\phi = a_3^2 e^\phi &= \frac{p_1^2 \cosh(p_2(\tau - \tau_0))}{p_2 \sinh^2(p_1 \tau))}, \\ a_2^2 e^\phi &= \frac{p_2}{\cosh(p_2(\tau - \tau_0)}, \end{aligned} \tag{50}$$

with the constants subject to the restriction

$$4p_1^2 - p_2^2 = A^2$$

as required by Eq. (23).

4.10 Type IX

Space-times of this type have been extensively studied in general relativity and they include the well-known Taub and mixmaster models [4],[17]. No analytic solution has been given for the latter case. In the former case the solution is obtained after the enlargement of the original isometry group to $SO(3) \times U(1)$. In the present context the field equations (22) are written in the general case as

$$(\ln a_i^2 e^\phi)'' + (a_i^4 - (a_j^2 - a_k^2)^2) e^{2\phi} = 0, \tag{51}$$

with (ijk) taken cyclically as (123) and there are no constraint equations. Eqs. (51) are precisely those of the mixmaster system which is known not to yield to an analytic treatment. Nevertheless, as in type VIII, under enlargment of the isometry to a G_4 (e.g., with $a_1 = a_3$) the system of equations (51) can be fully integrated. We have thus reproduced what may be considered as the string-cosmology analog of the Taub universe. This result may be expressed as

$$\begin{aligned} a_1^2 e^\phi = a_3^2 e^\phi &= \frac{p_1^2 \cosh(p_2(\tau - \tau_0))}{p_2 \cosh^2(p_1 \tau)}, \\ a_2^2 e^\phi &= \frac{p_2}{\cosh(p_2(\tau - \tau_0)}, \end{aligned} \tag{52}$$

where the constants p_1, p_2 are subject to the restriction

$$4p_1^2 - p_2^2 = A^2$$

as required by Eq. (23). Our solution reproduces Taub's metric if we switch off the H field ($A = 0$). It is interesting to note that, contrary to Taub's metric, the present generalization possesses a limit of complete isotropy, attainable at $p_1 = p_2 = A/\sqrt{3}$. We then obtain a closed (k=1) FRW model with radius

$$a^2 e^\phi = \frac{A}{\sqrt{3} \cosh(\frac{A}{\sqrt{3}} \tau)}, \tag{53}$$

which has already been discussed in the quoted literature.

5 Conclusions

We have explicitly found a general class of background space-times in the context of homogeneous string cosmology, with non-vanishing dilaton and antisymmetric H field. Our findings have been obtained under the assumption of vanishing cosmological constant and for metrics which are diagonal in the invariant $\{\sigma^i\}$ basis. It follows that (modulo the doubling of classes caused by a non-vanishing cosmological constant) there exists only one other general class in homogeneous string cosmology. It is that of metrics (11) which cannot be diagonalized in the above ivariant basis. The distinction between these two fundamental classes can be better illuminated on more physical grounds. We observe that, in the non-diagonal case, the gradient of the axion field will generally depend not only on t but on spatial coordinates as well. Equivalently (and disregarding here some marginal cases), the dual of H would no longer be orthogonal to the hypersurfaces of homogeneity Σ^3. One may now deduce that these 'tilted' models represent rotating universes [1],[5],[16]. In the respective space-times, cosmic vorticity will emerge in addition to expansion and shear, already existing in the case of diagonal metrics to which we now return. Explicit expressions for the standard parameters of expansion (Hubble constants) and shear (anisotropy) for all Bianchi types can be red-off (or easily found) directly from our results in section 4. All general solutions presented in that section are new and reduce to known results and isotropic limits as summarized in the Table. Following the reasoning in section 2, it may be expected that the respective models could illuminate the dynamics of the early universe, where string theory probably has its best (if not the only) chance of been confronted with reality.

The following may be noted as preliminaries to a more detailed study (e.g., in the approach utilized in [10] for the isotropic case). We firstly recall that our results may be equivalently expressed in the Einstein frame (now aptly in terms of the radii $\tilde{a}_i$ etc.) and examined there in a general relativistic context. We already have at out disposal the energy momentum tensors (21) for the ϕ and H fields which have the form of the energy-momentum tensor of a perfect fluid. Such an identification assigns a 'stiff matter' behaviour to these fields, while the associated $\rho = p$ equation of state does not upset the energy conditions.The former result reveals a gravity-like nature for the dilaton and H field (of importance when generating these fields from vacuum configurations), while the latter makes inevitable the presence of an initial singularity (which is indeed the case with all our solutions). On similar grounds one may generally verify that none of our solutions can have inflation. This follows from the sign of the quantity $\frac{d^2\tilde{a}}{d\tilde{t}^2}$, which is always negative - thus forbidding superluminal expansion. Also of apparent theoretical (and possibly observational as well) significance is the inter-relation between the Hubble constants along the principal directions of anisotropy and the magnitute of the H-field strength as required by the initial value equation (23). A particular example of this inter-

relation has been the existence of the $SO(3)$ isotropy limit in our generalised Taub metric (52). Indeed, at the $A = 0$ limit, the isotropic solution (53) collapses to a single singular point, in agreement with the non-existence of a full-isotropy limit in Taub's solution.

Obviously there are several further issues to be investigated in the more general context of string theory. We will only examine below whether new solutions can be generated from the above by duality transformations. As well known, target space duality is one of the most fundamental symmetries in string theory relating, as it does, small to large scales (roughly speaking a radius a_0 in the original metric to its inverse a_0^{-1} in the dual) [18],[19]. The calculations for abelian duality are straightforward but rather awkward to be explicitly presented here, the reason being that they had to be carried-out in holonomic coordinates, to which all our solutions were firstly re-expressed. Fortunately however, the final result (summarized in the Table) may be expressed quite concisely, at least for certain Bianchi types. These may be grouped as (I, II) and (VI_h) (with the h-parameter typically allowed hereafter to also take the values 0,1 for types III, V respectively). These types retain their homogeneity and in fact their duals reproduce metrics as those in the Bianchi-types (II, I) and (VI_{-h}) respectively in that order. The situation is considerably more complex for types $IV, VII_h, VIII, IX$ for which homogeneity seems to be generally lost under duality. However, when equipped with a G_4 symmetry, these spacetimes essentially reproduce themselves. In particular, this is precisely the case with our generalized-Taub solution which reproduces duals with the same type-IX metric.

We would like to thank I. Bakas for discussions.

NOTE ADDED:

Ref. [20], which appeared shortly after the completion of the present work addresses part of the same general problem. The Bianchi types examined there correspond to our types I, II and VI_h for vanishing H-field.

Appendix

Here we review the formalism and definitions needed for the reproduction of our results. Computations simplify considerably if carried out directly in the invariant basis $\{\sigma^i\}$ rather than in conventional holonomic coordinates. For even further simplification one may express (12) as a Minkowski metric, namely as

$$ds^2 = \eta_{\mu\nu}\omega^\mu\omega^\nu$$

with $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. In this orthonormal non-holonomic expression, which is defined in the sigma frame, the $\{\omega^\mu\}$ basis has been chosen as

$$\begin{aligned} \omega^0 &= dt, \\ \omega^i &= a_i(t)\sigma^i \ (no\ sum), \end{aligned} \tag{54}$$

We now recall that, for vanishing torsion, Cartan's first structure equation may be expressed as

$$d\omega^\mu = \gamma^\mu_{\nu\rho}\omega^\nu \wedge \omega^\rho. \tag{55}$$

The coefficients $\gamma^\mu_{\nu\rho}$ are the components of the connection 1-form ω^μ_ν, so that

$$\gamma_{\mu\nu\rho} = -\gamma_{\nu\mu\rho}.$$

The components of the Riemann tensor are now supplied by Cartan's second structure equation which may be here expressed as

$$\frac{1}{2}R^\lambda_{\mu\rho\nu}\omega^\rho \wedge \omega^\nu = d\omega^\lambda_\mu + \omega^\lambda_\rho \wedge \omega^\rho_\mu. \tag{56}$$

Hence, the components of the Ricci tensor may be expressed as

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \partial_\rho\gamma^\rho_{\mu\nu} - \partial_\nu\gamma^\rho_{\mu\rho} - \gamma^\rho_{\mu\lambda}\gamma^\lambda_{\nu\rho} + \gamma^\rho_{\lambda\rho}\gamma^\lambda_{\mu\nu}. \tag{57}$$

It is precisely the above components which have been utilised in writting down the gravitational field equations in section 3.2.

Bianchi-type classification

Type	Cl.	$d\sigma^i = \frac{1}{2}C^i_{jk}\sigma^j\sigma^k$	Coordinate basis	$a_i(t)$	FRW	Dual
I	A	$d\sigma^1 = 0$ $d\sigma^2 = 0$ $d\sigma^3 = 0$	$\sigma^i = dx^i$	(26)	YES	II
II	A	$d\sigma^1 = \sigma^2 \wedge \sigma^3$ $d\sigma^2 = 0$ $d\sigma^3 = 0$	$\sigma^1 = dx^2 - x^1 dx^3$ $\sigma^2 = dx^3$ $\sigma^3 = dx^1$	(29)	NO	I
III	B	$d\sigma^1 = 0$ $d\sigma^2 = 0$ $d\sigma^3 = \sigma^1 \wedge \sigma^3$	$\sigma^1 = dx^1$ $\sigma^2 = dx^2$ $\sigma^3 = e^{x^1} dx^3$	(33)	NO	III
IV	B	$d\sigma^1 = \sigma^1 \wedge \sigma^3 +$ $\sigma^2 \wedge \sigma^3$ $d\sigma^2 = \sigma^2 \wedge \sigma^3$ $d\sigma^3 = 0$	$\sigma^1 = e^{-x^1} dx^2 - x^1 e^{-x^1} dx^3$ $\sigma^2 = e^{-x^1} dx^3$ $\sigma^3 = dx^1$	*	NO	*
V	B	$d\sigma^1 = 0$ $d\sigma^2 = \sigma^1 \wedge \sigma^2$ $d\sigma^3 = \sigma^1 \wedge \sigma^3$	$\sigma^1 = dx^1$ $\sigma^2 = e^{x^1} dx^2$ $\sigma^3 = e^{x^1} dx^3$	(38)	YES	VI_{-1}
VI_h VI_{-1}	B	$d\sigma^1 = 0$ $d\sigma^2 = h\sigma^1 \wedge \sigma^2$ $d\sigma^3 = \sigma^1 \wedge \sigma^3$	$\sigma^1 = dx^1$ $\sigma^2 = e^{hx^1} dx^2$ $\sigma^3 = e^{x^1} dx^3$	(43) 46	NO	VI_{-h}
VII_h $h \neq 0$	B	$d\sigma^1 = -\sigma^2 \wedge \sigma^3$ $d\sigma^2 = \sigma^1 \wedge \sigma^3 +$ $h\sigma^2 \wedge \sigma^3$ $d\sigma^3 = 0$	$\star$ $\sigma^1 = (A - kB)dx^2 - Bdx^3$ $\sigma^2 = Bdx^2 + (A + kB)dx^3$ $\sigma^3 = dx^1$	*	NO	*
VII_0	A			** (48)	YES	I
VIII	A	$d\sigma^1 = \sigma^2 \wedge \sigma^3$ $d\sigma^2 = -\sigma^3 \wedge \sigma^1$ $d\sigma^3 = \sigma^1 \wedge \sigma^2$	$\sigma^1 = dx^1 + ((x^1)^2 - 1)dx^2 +$ $(x^1 + x^2 - x^2(x^1)^2)dx^3$ $\sigma^2 = dx^1 + ((x^1)^2 + 1)dx^2 +$ $(x^1 - x^2 - x^2(x^1)^2)dx^3$ $\sigma^3 = 2x^1 dx^2 + (1 - 2x^1 x^2)dx^3$	? ** (50)	 NO	? VIII
IX	A	$d\sigma^1 = \sigma^2 \wedge \sigma^3$ $d\sigma^2 = \sigma^3 \wedge \sigma^1$ $d\sigma^3 = \sigma^1 \wedge \sigma^2$	$\sigma^1 = -\sin x^3 dx^1 +$ $\sin x^1 \cos x^3 dx^2$ $\sigma^2 = \cos x^3 dx^1 +$ $\sin x^1 \sin x^3 dx^2$ $\sigma^3 = \cos x^1 dx^2 + dx^3$	? ** (52)	 YES	? IX

* Solutions singular everywhere.

** Solution given for $a_1 = a_3$.

* $A = e^{-kx^1}\cos(qx^1),\ B = -\frac{1}{q}e^{-kx^1}\sin(qx^1), \quad [k = \frac{h}{2},\ q = \sqrt{1-k^2}]$

References

[1] M.A.H. MacCallum, *"Anisotropic and Inhomogeneous Relativistic Cosmologies"*, in *"General Relativity-An Einstein Centenary Survey"*, eds. S. W. Hawking and W. Israel, Cambridge Univ. Press, Campridge, 1979:
S. Barrow and D.H. Sonoda, Phys. Rep. 139 (1986)1.

[2] M. Muller, Nucl. Phys. B337 (1990)37.

[3] E.J. Copeland, A. Lahiri and D. Wands, Phys. Rev. D50 (1994)4868; *"Srting cosmology with a time-dependent antisymmetric-tensor potential"* preprint Susssex-Ast/94/10-1, hep/th/9410136.

[4] A. Taub, Ann. Math. 53 (1951)472.

[5] G.F.R. Ellis and M.A.H. MacCallum, Commun. Math Phys. 12 (1969)108; 19 (1970)31.

[6] E. Fradkin and A. Tseytlin, Nucl. Phys. B261 (1985)1;
C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. B262 (1985)593.

[7] I. Bakas, Nucl. Phys. B 428 (1994)374; *"Space-time interpretation of S-duality and supersymmetry violation of T-duality"*, preprint CERN-th.7473/94, hep-th/9410104.

[8] N. Sanchez and G. Veneziano, Nucl. Phys. B333 (1990)253;
B.A. Campbell, A. Linde and K.A. Olive, Nucl. Phys. B355 (1991)146.

[9] R. Myers, Phys. Lett. B199 (1987)371;
K.A. Meissner and G. Veneziano, Phys. Lett. B267(1991)33;
A. Sen, Phys. Lett. B271 (1991)295;
M.Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B272 (1991)277;
M. Gasperini and G. Veneziano, Phys. Lett. B277 (1992)256; Mod. Phys. Lett. A8 91993)3701; M. Gasperini, R. Ricci and G. Veneziano, Phys. Lett. B319 (1993)438;
A.A. Tseytlin, Class Quantum Grav. 9 (1992)979.

[10] R. Brandenberger and C. Vafa, Nucl. Phys. B316 (1988)319;
A.A. Tseytlin and C. Vafa, Nucl. Phys. B372 91992)443;

[11] I. Antoniadis, C. Bachas, J. Ellis and D.V. Nanopoulos, Phys. Lett. B211 (1988)393; Nucl. Phys. B328 (1989)117.

[12] C. Kounnas and D. Lüst, Phys. Lett. B 289 (1992)56.

[13] C. Nappi and E. Witten, Phys. Lett. B 293 (1992)309.

[14] A. Giveon and A. Pasquinucci, Phys. Lett. B294(1992)162;
A. Giveon and M. Rocek, Nucl. Phys. B 380 (1992)128;
P. Horova, Phys. Lett. B 278 (1992)101.

[15] E.W. Kolb and M.S. Turner, *"The Early Universe: Reprints"*, Addison-Wisley, N.Y., 1988; *"The Early Universe"*, Addison-Wisley, N.Y., 1990.

[16] M. Ryan and L. Shepley, *"Homogeneous Relativistic Cosmologies*, Princeton Univ. Press, Princeton, 1975:
D. Kramer *et. al.*, *"Exact solutions of Einstein's field equations"*, Cambridge Univ. Press, Cambridge, 1980.

[17] C.W. Misner, Phys. Rev. Lett. 22 (1969)1071.

[18] K. Kikkawa and M. Yamasaki, Phys. Lett. B 149 (1984)357;
N. Sakai and I. Senda, Prog. Theo. Phys. 75 (1984)692;
N. Narain, Phys. Lett. B 169 (1986)41:
T. H. Buscher, Phys. Lett. B 194 (1987)51;
A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B 322 (1989)167.

[19] E. Kiritsis, Mod. Phys. Lett A6 (1991)2871; Nucl. Phys. B 405 (1993)109;
M. Rocek and E. Verlinde, Nucl. Phys. B 373 (1992)630;
A. Giveon and M. Rocek, Nucl. Phys. B 380 (1992)128;
A. Giveon and E. Kiritsis, Nucl. Phys. B 411 (1991)487;
A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 224 (1994)77;
E. Alvarez, L. Alvarez-Gaumé and Y. Lozano, *"On non-abelian duality*, Preprint CERN-TH.7204/94.

[20] M. Gasperini and R. Ricci, *"Homogeneous conformal string backgrounds"*, preprint DFTT-02/95, hep-th/9501055.